Zoom Class 8/5/2020

Ques Evaluate $\displaystyle\int_0^{\pi/2} \frac{\cos x\, dx}{(1+\sin x)(2+\sin x)}$

Soln $\because$ Numerator has $\cos x$ which is differential of $\sin x$

we substitute $\sin x = y$

$\Rightarrow \cos x\, dx = dy$

And when $x \to 0$, $y \to \overset{\sin}{\cancel{\cos}}\, 0 = 0$

when $x \to \pi/2$, $y \to \overset{\sin}{\cancel{\cos}}\, \pi/2 = 1$

$$\therefore I = \int_0^1 \frac{dy}{(1+y)(2+y)}$$

$$= \int_0^1 \frac{\{(2+y)-(1+y)\}\,dy}{(1+y)(2+y)}$$

$$= \int_0^1 \frac{dy}{(1+y)} - \int_0^1 \frac{dy}{(2+y)}$$

$$= \Big[\log(1+y)\Big]_0^1 - \Big[\log(2+y)\Big]_0^1$$

$$= \log 2 - \log 1 - \log 3 + \log 2$$

$$= \log 4 - \log 3 \quad (\because \log mn = \log m + \log n)$$

$$= \log\left(\frac{4}{3}\right) \text{ —— Ans} \quad \left(\because \log\frac{m}{n} = \log m - \log n\right)$$

Ques Evaluate $\int_{\alpha}^{\beta} \sqrt{(x-\alpha)(\beta-x)}\, dx$

Soln Putting $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$\Rightarrow dx = 2\alpha\cos\theta(-\sin\theta)d\theta + 2\beta\sin\theta\cos\theta d\theta$
$= 2\cos\theta\sin\theta(\beta-\alpha)d\theta$ —— (1)

Also $x-\alpha = \alpha\cos^2\theta + \beta\sin^2\theta - \alpha$
We try to convert above equation in form of eqn (1)

$\Rightarrow x-\alpha = \beta\sin^2\theta + \alpha(\cos^2\theta - 1)$
$= \beta\sin^2\theta - \alpha\sin^2\theta$
$= \sin^2\theta(\beta-\alpha)$ —— (2)

Similarly,

$(\beta-x) = -(x-\beta) = -(\alpha\cos^2\theta + \beta\sin^2\theta - \beta)$
$= -(\alpha\cos^2\theta + \beta(\sin^2\theta - 1))$
$= -(\alpha\cos^2\theta - \beta\cos^2\theta)$
$= -\cos^2\theta(\alpha-\beta) = \cos^2\theta(\beta-\alpha)$ —— (3)

Also when $x \to \alpha$,
$\alpha\cos^2\theta + \beta\sin^2\theta = \alpha \Rightarrow \alpha\cos^2\theta - \beta\sin^2\theta - \alpha = 0$
$\Rightarrow (\beta-\alpha)\sin^2\theta = 0$ —— (from (2))
$\Rightarrow \sin^2\theta = 0 \Rightarrow \sin\theta = 0$
$\Rightarrow \theta = 0$ {if $\alpha \neq \beta$}

and when $x \to \beta$
$\alpha\cos^2\theta + \beta\sin^2\theta = \beta \Rightarrow \alpha\cos^2\theta + \beta\sin^2\theta - \beta = 0$
$\Rightarrow -\{\alpha\cos^2\theta + \beta\sin\theta - \beta\} = 0$
$\Rightarrow (\beta-\alpha)\cos^2\theta = 0$ —— {from (3)}

$$\Rightarrow \cos^2\theta = 0 \longrightarrow \{\text{if } \alpha \neq \beta\}$$

$$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \pi/2$$

$$\therefore I = \int_0^{\pi/2} \sqrt{(\beta-\alpha)\cos^2\theta\,(\beta-\alpha)\sin^2\theta}\,.\ 2(\beta-\alpha)\sin\theta\cos\theta\, d\theta$$

$$= 2(\beta-\alpha)\int_0^{\pi/2}(\beta-\alpha)\cos\theta\sin\theta\,.\,\sin\theta\cos\theta\, d\theta$$

$$= 2(\beta-\alpha)^2\int_0^{\pi/2}\cos^2\theta\sin^2\theta\, d\theta$$

$$= \frac{1}{2}(\beta-\alpha)^2\int_0^{\pi/2} 4\cos^2\theta\sin^2\theta\, d\theta$$

$$= \frac{1}{2}(\beta-\alpha)^2\int_0^{\pi/2}(2\cos\theta\sin\theta)^2\, d\theta$$

$$= \frac{1}{2}(\beta-\alpha)^2\int_0^{\pi/2}(\sin 2\theta)^2\, d\theta.$$

$$= \frac{1}{2}(\beta-\alpha)^2\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right) d\theta$$

$$= \frac{1}{4}(\beta-\alpha)^2\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{1}{4}(\beta-\alpha)^2\left[\frac{\pi}{2} - \frac{\sin 2\pi}{4} - 0 + \frac{\sin 0}{4}\right]$$

$$= \frac{1}{8}(\beta-\alpha)^2\left[\pi - 0 - 0 + 0\right]$$

$$= \frac{\pi}{8}(\beta-\alpha)^2 \quad \underline{\text{Ans}}$$

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Ques Evaluate $\int_0^{\pi/2} \frac{\sin 2\theta}{\sin^4\theta + \cos^4\theta} d\theta$

Soln $I = \int_0^{\pi/2} \frac{\sin 2\theta}{\sin^4\theta + \cos^4\theta} d\theta$

$\{\because a^2 + b^2 = (a+b)^2 - 2ab\}$

$$I = \int_0^{\pi/2} \frac{\sin 2\theta \, d\theta}{(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta}$$

$$= \int_0^{\pi/2} \frac{\sin 2\theta \, d\theta}{1 - 2\sin^2\theta\cos^2\theta}$$

$$= \int_0^{\pi/2} \frac{2\sin\theta\cos\theta \, d\theta}{1 - 2\sin^2\theta(1 - \sin^2\theta)}$$

$$= \int_0^{\pi/2} \frac{2\sin\theta\cos\theta \, d\theta}{1 - 2\sin^2\theta + 2\sin^4\theta}$$

Put $\sin^2\theta = x \Rightarrow 2\sin\theta\cos\theta \, d\theta = dx$

If $\theta \to 0 \Rightarrow x = \sin^2 0 = 0$

if $\theta \to \frac{\pi}{2} \Rightarrow x = \left(\sin\frac{\pi}{2}\right)^2 = 1^2 = 1$

$$\therefore I = \int_0^1 \frac{dx}{1 - 2x + 2x^2} = \frac{1}{2}\int_0^1 \frac{dx}{x^2 - x + 1/2}$$

$$= \frac{1}{2}\int_0^1 \frac{dx}{x^2 - 2 \cdot x \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$ $\{\frac{1}{2}$ is divided into $\frac{1}{4} + \frac{1}{4}$

$$\Rightarrow I = \frac{1}{2}\int_0^1 \frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{2}\left[\frac{1}{1/2}\tan^{-1}\left(\frac{x-1/2}{1/2}\right)\right]_0^1$$

$$= \frac{1}{\cancel{2}}\times\cancel{2}\left[\tan^{-1}\left\{\frac{(2x-1)\times\cancel{2}}{\cancel{2}}\right\}\right]_0^1$$

$$= \left[\tan^{-1}(2x-1)\right]_0^1$$

$$= \tan^{-1}(2-1) - \tan^{-1}(0-1)$$

$$= \tan^{-1}1 - \tan^{-1}(-1)$$

$$= \tan^{-1}(1) + \tan^{-1}(1)$$

$$= 2\tan^{-1}1 = 2\times\frac{\pi}{\cancel{4}2} = \frac{\pi}{2} \text{ Ans}$$

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Ques Evaluate $\int_0^{\pi/2} \frac{x^2\,dx}{(x\sin x+\cos x)^2}$

Soln $I = \int_0^{\pi/2} \frac{x^2\,dx}{(x\sin x+\cos x)^2}$

If we diffrentiate $\frac{1}{x\sin x+\cos x}$ using the we get $\frac{-1}{(x\sin x+\cos x)^2}\times(x\cos x+\sin x-\sin x)$

$$= \frac{-x\cos x}{(x\sin x+\cos x)^2}$$

So transforming the integral we get

$$I = \int_0^{\pi/2} \frac{x}{\cos x}\cdot\frac{(-x\cos x)\,dx}{(x\sin x+\cos x)^2}$$

Using integration by parts without limits we get

$$I = -\left[\frac{x}{\cos x}\left(\frac{1}{x\sin x+\cos x}\right) - \int\left(\frac{\cos x + x\sin x}{\cos^2 x}\right)\left(\frac{1}{x\sin x+\cos x}\right)dx\right]$$

$$= -\left[\frac{x}{\cos x(x\sin x+\cos x)} - \int\frac{dx}{\cos^2 x}\right]$$

$$= \left[\frac{-x}{\cos x(x\sin x+\cos x)} + \int\sec^2 x\,dx\right]$$

$$= \left[\frac{-x}{\cos x(x\sin x+\cos x)} + \tan x\right]$$

$$= \frac{\sin x(x\sin x+\cos x) - x}{\cos x(x\sin x+\cos x)}$$ Putting $\tan x = \frac{\sin}{\cos}$ & taking LCM

$$= \frac{x\sin^2 x + \sin x\cos x - x}{\cos x(x\sin x+\cos x)} = \frac{\sin x\cos x - x(1-\sin^2 x)}{\cos x(x\sin x+\cos x)}$$

$$= \frac{\sin x \cos x - x\cos^2 x}{\cos x\,(x\sin x + \cos x)}$$

$$= \frac{\cancel{\cos x}(\sin x - x\cos x)}{\cancel{\cos x}\,(x\sin x + \cos x)}$$

$$\Rightarrow I = \frac{\sin x - x\cos x}{x\sin x + \cos x}$$

Now Putting the limits we get

$$I = \int_0^{\pi/2} \frac{x^2\,dx}{(x\sin + \cos x)^2} = \left[\frac{\sin x - x\cos x}{x\sin x + \cos x}\right]_0^{\pi/2}$$

$$= \left[\frac{\sin \pi/2 - \frac{\pi}{2}\cos \pi/2}{\pi/2 \sin \pi/2 + \cos \pi/2} - 0\right]$$

∵ Numerator becomes zero

03 THURSDAY

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$$= \frac{(1 - \pi/2 \cdot 0)}{\frac{\pi}{2} 1 + 0} = \frac{1-0}{\pi/2}$$

$$= \frac{2}{\pi}$$

Ans